THEORY OF A TURBULENT VORTEX JET OF INCOMPRESSIBLE FLUID

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An approximate solution is given for the equations of motion and energy in the main part of a turbulent vortex jet of incompressible fluid in a space full of the same fluid.

If a turbulent vortex jet discharges at a sufficiently large Re number, its radial velocity component will be very small in comparison with the axial and tangential components. The jet may then be considered as a motion possessing the properties of a boundary layer, so that the Navier-Stokes and energy equations describing the total variation of the parameters of the jet can be simplified.

In cylindrical coordinates, the boundary layer equations of turbulent motion of a viscous and thermally conducting fluid in a vortex jet (all derivatives with respect to the angle θ are zero due to the axial symmetry of the jet) with Pr = 1 will have the form

$$\begin{gathered}\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}\right)=-\frac{\partial p}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(r,\ \tau_{xr}),\\ \rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial r}+\frac{vw}{r}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{x\theta}),\\ \frac{\partial p}{\partial r}=\frac{\rho w^2}{r},\quad \frac{\partial(\rho ur)}{\partial x}+\frac{\partial(\rho vr)}{\partial r}=0,\\ \rho c_p\left(u\frac{\partial\Delta T}{\partial x}+v\frac{\partial\Delta T}{\partial r}\right)=\frac{1}{r}\frac{\partial}{\partial r}(rq).\end{gathered}\tag{1}$$

In accordance with boundary layer theory [1], we neglect the influence of molecular viscosity and omit the normal components of the turbulent stress tensor. We also bear in mind that the axial derivatives are small in comparison with the radial.

Using the method of successive approximations to solve (1), we can obtain formulas for u, v, w, and p in the main part of a turbulent jet of incompressible fluid discharging into a space full of the same fluid [2]. The mathematical operations are very tedious, however, and difficult to apply to the investigation of high-temperature and high-speed turbulent vortex jets. Transforming the second equation of (1) by making the substitution M = wr reduces the differential equations of motion and energy of the boundary layer of the jet to forms that can be analyzed without expanding the components of the averaged velocity, temperature, and pressure.

Consider a turbulent vortex jet of incompressible fluid. From the hypothesis that the turbulent transfer coefficient is constant over the cross section of the jet, the tangential components of the turbulent stress tensor may be written in the form

$$\tau_{xr}=\rho\varepsilon\frac{\partial u}{\partial r},\quad \tau_{x\theta}=\rho\varepsilon\frac{1}{r}\left(\frac{\partial M}{\partial r}-\frac{2M}{r}\right).\tag{2}$$

Now the equations of (1) will be analogous to those for a laminar vortex jet of incompressible fluid with the dynamic viscosity μ replaced by ε. Then the solution of the first equation of (1), neglecting the influence of pressure on the axial velocity component, will have the form [2, 3]:

$$\frac{u}{u_m}=\frac{F'(\xi)}{\xi}=\frac{1}{(1+0.125\,\xi^2)^2}.\tag{3}$$

Substituting M = wr in the second equation of (1), we have

$$u\frac{\partial M}{\partial x}+v\frac{\partial M}{\partial r}=\frac{\varepsilon}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial M}{\partial r}-\frac{2M}{r}\right)\right].\tag{4}$$

We shall seek a solution in the form

$$M = a(\xi)/x. \tag{5}$$

Integrating (4) and taking into account (3) and (5), we obtain

$$\frac{\partial a(\xi)}{a(\xi)} = \frac{2 - F(\xi)}{\xi}\partial\xi. \tag{6}$$

The constant of integration in this equation is zero, since $a(\xi) = 0$ when $\xi = 0$.

Substituting the value of $F(\xi)$ into (6), after integration we have the following expression for M:

$$M = c_1\xi^2/x(1 + 0.125\xi^2)^2. \tag{7}$$

If we assume that the solution for p has the form $p = b(\xi)/x^4$, then, integrating the third equation of (1), for the pressure profile in the jet we have

$$p/p_m = (1 + 0.125\xi^2)^{-3}. \tag{8}$$

Expressions (7) and (8) are analogous to the formulas for w and p obtained in investigating a turbulent vortex jet by the method of successive approximations.

To determine the dimensionless excess temperature in the main part of the turbulent vortex jet we have the relation [3]:

$$\Delta T/\Delta T_m = \sqrt{u/u_m}\,. \tag{9}$$

If we consider a vortex jet of finite thickness, the hypothesis that the turbulent transfer is constant does not give the desired result, especially at the boundaries of the jet.

An axial velocity component distribution that agrees well with experimental data [3] over the whole width of the jet is obtained by putting

$$\tau_{xr} = \rho l_T^2 \left|\frac{\partial u}{\partial r}\right| \frac{\partial u}{\partial r}, \qquad \tau_{x\theta} = \rho l_T^2 \left|\frac{\partial u}{\partial r}\right| \left(\frac{\partial w}{\partial r} - \frac{w}{r}\right). \tag{10}$$

In this case integration of the second equation of (1) gives

$$\frac{\partial a(\varphi)}{a(\varphi)} = \frac{F(\varphi) + 2[F'(\varphi)/\varphi]'}{\varphi[F'(\varphi)/\varphi]'}\partial\varphi.$$

A numerical integration of the function

$$a(\varphi) = c_2\varphi^2 \exp\int_0^{\varphi_b} \frac{F(\varphi) + 2[F'(\varphi)/\varphi]'}{\varphi[F'(\varphi)/\varphi]'}\partial\varphi \tag{11}$$

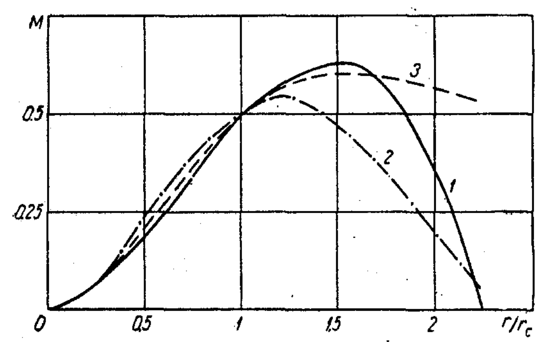

Fig. 1. Moment fields of tangential velocity components in the main part of a turbulent vortex jet of incompressible fluid ($c_1/x = 0.392 \cdot$ $\cdot\, c_2/x = 0.1075$ $c_3/x = 1$): 1 – according to (13), 2 – (11), 3 – (7)

for an axial velocity component distribution determined by the Tollmien solution [3] is given in Fig. 1.

The solution of the first equation (1) for u may be expressed with sufficient accuracy by the equation

$$u/u_m = (1 - \varphi^{\frac{3}{2}})^2. \tag{12}$$

In this case the solution of the second equation of system (1) is given by the formula

$$M = \frac{c_3}{x}(1 - \varphi^{\frac{3}{2}})\varphi^{\frac{7}{4}}, \tag{13}$$

where c_3, as well as c_1 and c_2, is determined from the condition that L_0 is constant in the jet. However, since the axial velocity component distributions in (3), (12) and in the Tollmien solution are described by different formulas, the numerical value of c_3 will not be equal to c_1 and c_2. Comparing c_1, c_2 and c_3 for $r/r_c = 1$ and $M = 0.5$, where r_c is the ordinate of the point at which the axial velocity component is half u_m, we obtain $c_1 = 0.392c_2 = 0.1075c_3$.

It can be seen from Fig. 1 that (7) and (11) give the distribution $M = f(r/r_c)$ at the axis of the jet with approximately the same accuracy as for the axial velocity component profiles, but diverge at the jet boundaries. The curve of the moment of the tangential velocity, expressed by (13), gives approximately the average of (7) and (11).

In a turbulent jet of finite thickness, the pressure distribution may be represented by

$$p/p_m = (1 - \bar{\varphi}^{\frac{3}{2}})^3.$$

Figure 2 shows the pressure over the jet cross section as a fraction of the value at the axis, as determined from (8) and (14). As with u and M, the curves diverge only at the jet boundary. Since the thermal and dynamic boundary layers do not interact in a flow of incompressible fluid, the distribution of dimensionless excess temperature in a vortex jet of finite thickness may also be described with the aid of (9), by substituting values of u determined from (12) or Tollmien's solution.

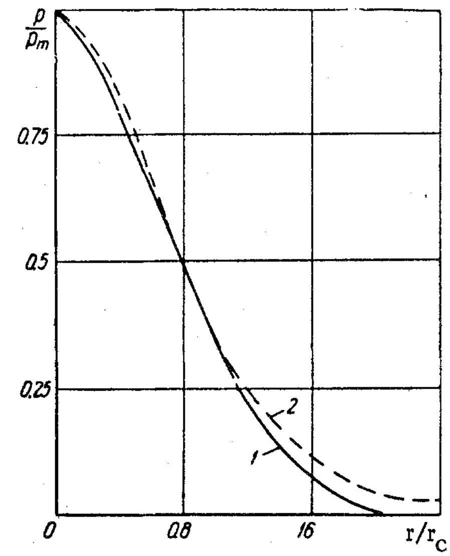

Fig. 2. Pressure profiles in the main part of a turbulent vortex jet: 1 – according to the formula $p/p_m = (1 - \bar{\varphi}^{3/2})^3$, $\bar{\varphi} = 0.441\, r/r_c$; 2 – $p/p_m = [1/(1 + 0.125\xi^2)]^3$, $\xi = 1.81\, r/r_c$

NOTATION

u, v and w – axial, radial, and tangential components of mean velocity of jet; T, T_a – temperature of fluid in jet and in surrounding medium; $\Delta T = (T - T_a)$ – excess temperature of fluid in jet; A – mechanical equivalent of heat; c_p – heat capacity of fluid; p – pressure of fluid in jet; ρ – density of fluid; q – heat flux; τ_{xr}, $\tau_{x\theta}$ – tangential components of turbulent stress tensor; ε – coefficient of turbulent transfer; u_m – axial velocity component at axis of jet; $\xi = r/\sigma x$, $\varphi = r/\varkappa x$, $\bar{\varphi} = r/\bar{\varkappa} x$ – relative coordinates; σ – an experimental constant; $F(\xi) = 0.5\,\xi/(1 + 0.125\xi^2)$ – function of ξ in the Hertler solution [3]; $a(\xi)$ and $b(\xi)$ – unknown functions of ξ; c_1 – a constant, determined from the condition that the moment of momentum in the jet is constant, i.e., for $L_0 = \int_0^\infty u\,Mrdr = \text{const}$; $\Delta T_m = T_m - T_a$ – excess temperature of fluid at axis of jet; l_T – Taylor mixing length; $\bar{\varkappa}$ – experimental constant, in general different from σ and x.

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